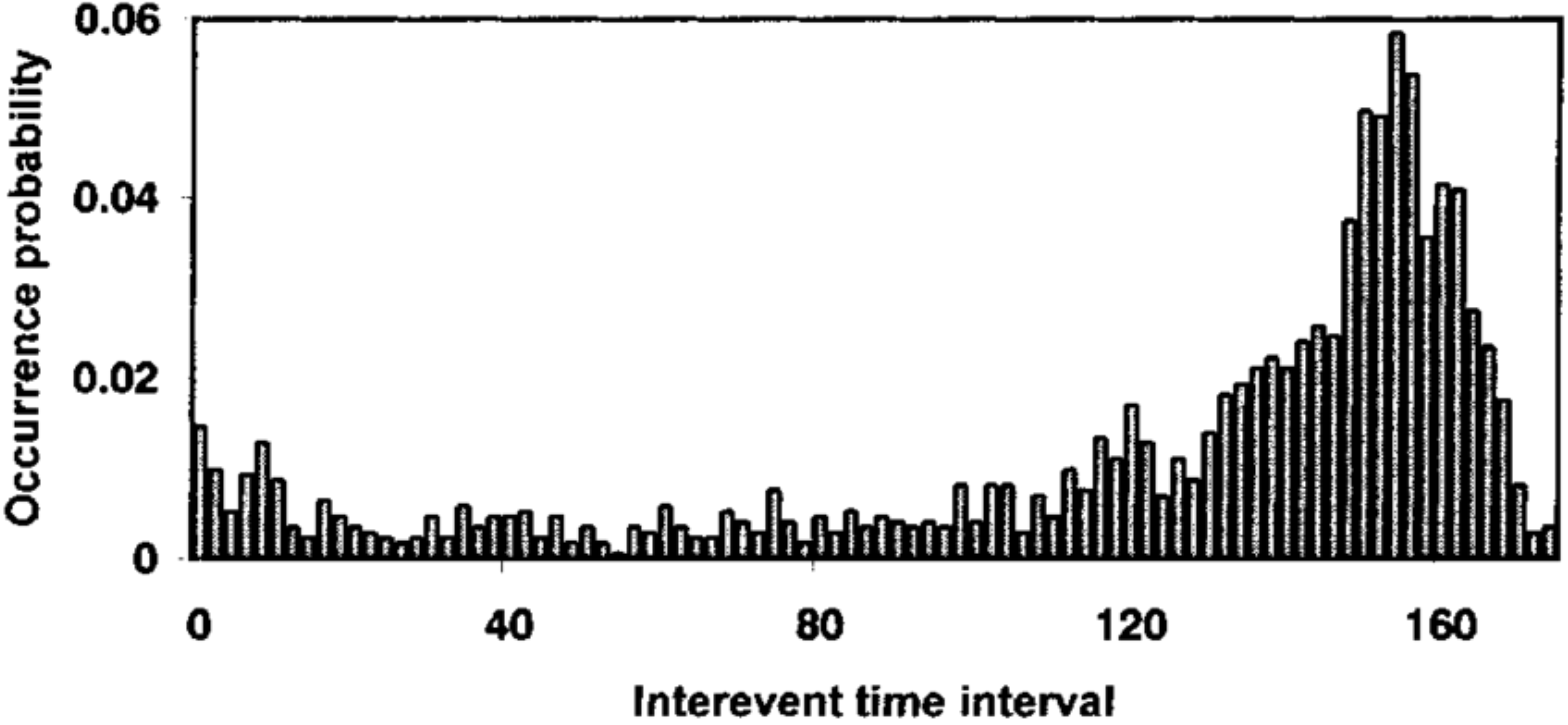
Lesson 019
The Exponential and Gamma Distributions

Wednesday, October 25

Not every continuous distribution is symmetric.



0.06
0.04
0.02
0
Occurrence probability
0
40
80
120
160
Interevent time interval

The Exponential Distribution

- A heavily skewed quantity is often represented using the **exponential distribution.**
- For instance: time between events, lifetimes of machines or components, magnitudes of earthquakes, insurance claim amounts, etc.
- The exponential distribution is characterized by a single parameter, λ.
  - This is called the **rate parameter** and is thought of as the **rate of event occurrence.**

The Exponential Distribution

- The PDF of the exponential is given by:

$$f(x) = \lambda e^{-\lambda x}, \quad x \geq 0$$

- The CDF of the exponential is given by:

$$F(x) = 1 - e^{-\lambda x}, \quad x \geq 0$$

- We also have $E[X] = \dfrac{1}{\lambda}$ and var$(X) = \dfrac{1}{\lambda^2}$.

The distribution of stress range in a certain bridge connection has an exponential distribution with a mean of 6 (MPa). What is the probability that the stress range is at most 10 MPa?

$1 - e^{-6(10)}$.

0%

$6e^{-6(10)}$.

0%

$1 - e^{-10/6}$.

0%

$\frac{1}{6}e^{-10/6}$.

0%

The distribution of stress range in a certain bridge connection has an exponential distribution with a mean of 6 (MPa). What is the probability that the stress range falls between 5 and 10 MPa?

$\frac{1}{6}e^{-10/6} - \frac{1}{6}e^{-5/6}$

0%

$e^{-10/6} - e^{-5/6}$

0%

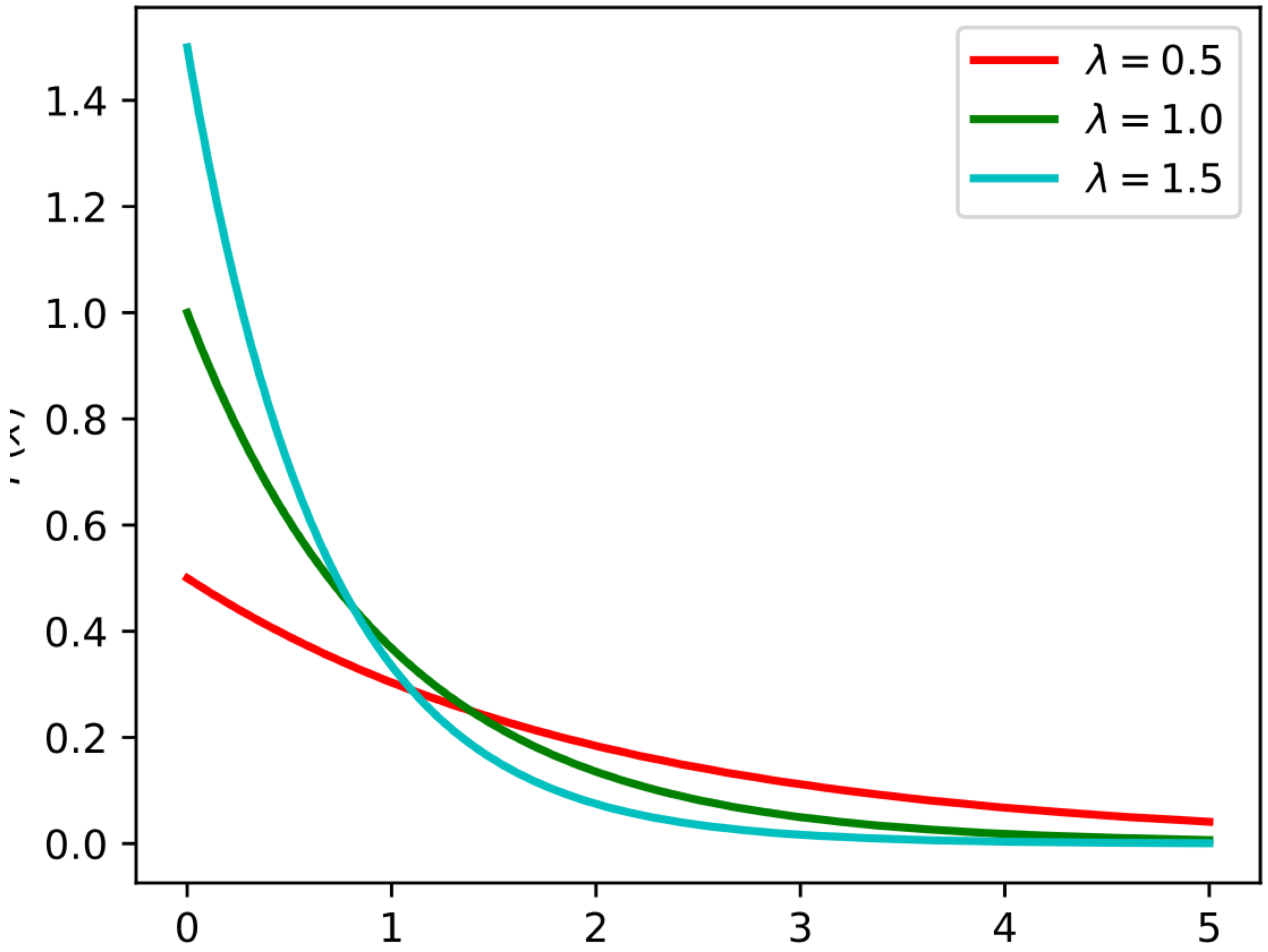
$e^{-5/6} - e^{-10/6}$

0%

$1 - e^{-10/6} - e^{-5/6}$

0%

The Exponential Distribtuion, Visually



The Memoryless Property

- An exponential random variable is said to be **memoryless.**

$$P(X \geq a \,|\, X \geq b) = P(X \geq a - b)$$

- Knowing that an exponential random variable exceeds a threshold does not change our understanding of its future behaviour.

Example

- Suppose that the number of kilometres that a car drives before engine failure follows an exponential distribution, with rate $\lambda = 0.000002$.
- **How many kilometres do we expect the car to drive?**
- **Given that the car has driven for $250{,}000$km already, how much further do we expect it to drive?**

The Exponential and the Poisson Process

- If events occur at a rate of α per unit time, we saw that their count on an interval of length t follows a Poi(αt) distribution.
- The time **between** successive events will follow an exponential distribution, with rate α.

Calls to a crisis center occur according to a Poisson process with rate $\alpha = 0.5$ calls per day. The crisis center is monitored for 10 days to understand the frequency of calls. What is the expected amount of time between successive calls?

0.5

0%

$\frac{1}{0.5} = 2.$

0%

$0.5(10) = 5$

0%

$\frac{1}{0.5(10)} = 0.2$

0%

The Gamma Distribution

- The exponential is a special case of the **gamma distribution**.
- The gamma distribution has two parameters, α and β, called the shape and scale parameters respectively.
- We have that $E[X] = \alpha\beta$ and $\text{var}(X) = \alpha\beta^2$.
- The PDF of the distribution relies on the **gamma function.**

The Gamma Function

- The gamma function is a function defined via an integral.

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx$$

- The gamma function generalizes factorials

$$\Gamma(n) = (n-1)! \quad n \in \mathbb{N}$$

$$\Gamma(\alpha) = (\alpha - 1) \cdot \Gamma(\alpha - 1)$$

$$\Gamma(0.5) = \sqrt{\pi}$$

The Gamma Distribution

- The PDF of the gamma distribution is

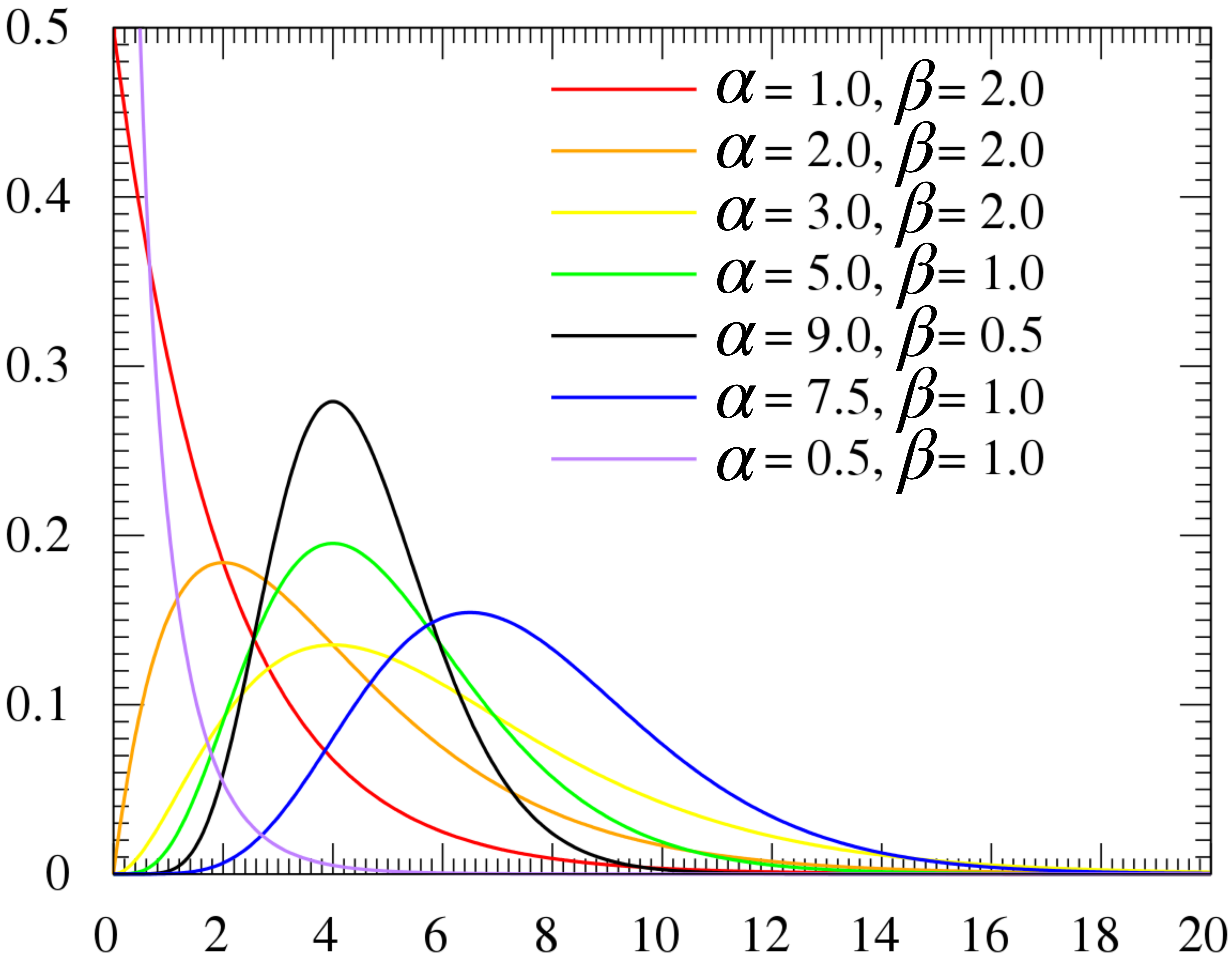
$$f(x) = \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} \exp\left(-\frac{x}{\beta}\right)$$

- The CDF of the gamma distribution is

$$F(x) = \frac{1}{\Gamma(\alpha)} \gamma\left(\alpha, \frac{x}{\beta}\right)$$

- Here, $\gamma(\alpha, x) = \int_0^x x^{\alpha-1} e^{-x} dx$ is the **incomplete gamma function.**

The Gamma Distribtuion, Visually



Suppose that the survival time, X, in weeks of a randomly selected mouse exposed to 240 rads of gamma radiation follows a $\mathrm{Gamma}(8, 15)$. What is the expected survival time?

8

0%

15

0%

$8 \times 15 = 120$

0%

$8 \times 15^2 = 1800$

0%

Relation to Other Distributions

- Setting $\beta = 1$ gives the **standard gamma distribution**.
- The Exponential distribution is a Gamma $\left(1, \frac{1}{\lambda}\right)$ distribution.
- If we take $\alpha = \frac{\nu}{2}$ and $\beta = 2$, this is called a **chi-square distribution.**
  - Denoted χ^2_ν, this will become important later.
  - If $Z \sim N(0,1)$, then $Z^2 \sim \chi^2_1$.